A Note on Closed Geodesics for a Class of non–compact Riemannian Manifolds

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1 Introduction

This paper is concerned with the existence of closed geodesics on a non–compact manifold M. There are very few papers on such a problem, see [3, 13, 14]. In particular, Tanaka deals with the manifod $M = \mathbb{R} \times S^N$, endowed with a metric $g(s,\xi) = g_0(\xi) + h(s,\xi)$, where g_0 is the standard product metric on $\mathbb{R} \times S^N$. Under the assumption that $h(s,\xi) \to 0$ as $|s| \to \infty$, he proves the existence of a closed geodesic, found as a critical point of the energy functional

$$E(u) = \frac{1}{2} \int_0^1 g(u)[\dot{u}, \dot{u}]\, dt, \tag{1}$$

defined on the loop space $\Lambda = \Lambda(M) = H^1(S^1, M)$ †. The lack of compactness due to the unboundedness of M is overcome by a suitable use of the concentration–compactness principle. To carry out the proof, the fact that M has the specific form $M = \mathbb{R} \times S^N$ is fundamental, because this permits to compare E with a *functional at infinity* whose behavior is explicitely known.

In the present paper, we consider a perturbed metric $g_\varepsilon = g_0 + \varepsilon h$, and extend Tanaka's result in two directions. First, we show the existence of at least N, in some cases $2N$, closed geodesics on $M = \mathbb{R} \times S^N$, see Theorem 2.6. Such a theorem can also be seen as an extension to cilyndrical domains of the result by Carminati [6]. Next, we deal with the case in which $M = \mathbb{R} \times M_0$ for a general compact N–dimensional manifold M_0 ‡. The existence result we are able to prove requires that either M_0 possesses a non–degenerate closed geodesic, see Theorem 3.5, or that $\pi_1(M_0) \neq \{0\}$ and the geodesics on M_0 are *isolated*, see Theorem 4.3.

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†We will identify S^1 with $[0,1]/\{0,1\}$.

‡By manifold we mean a smooth, connected manifold.

The approach we use is different that Tanaka's one, and relies on a perturbation result discussed in [1] that leads to rather simple proofs. Roughly, the main advantages of using this abstract perturbation method are that

(i) we can obtain sharper results, like the multeplicity ones;

(ii) we can deal with a general manifold like $M = \mathbb{R} \times M_0$, not only $M = \mathbb{R} \times S^N$, when the results – for the reasons indicated before – cannot be easily obtained by using Tanaka's approach.

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2 Spheres

In this section we assume $M = \mathbb{R} \times S^N$, where $S^N = \{\xi \in \mathbb{R}^{N+1} \colon |\xi| = 1\}$ [1]. For $s \in \mathbb{R}$, $r \in T_s\mathbb{R} \approx \mathbb{R}$, $\xi \in S^N$, $\eta \in T_\xi S^N$, let

$$g_0(s,\xi)((r,\eta),(r,\eta)) = |r|^2 + |\eta|^2 \tag{2}$$

be the standard product metric on $M = \mathbb{R} \times S^N$. We consider a perturbed metric

$$g_\varepsilon(s,\xi)((r,\eta),(r,\eta)) = |r|^2 + |\eta|^2 + \varepsilon h(s,\xi)((r,\eta),(r,\eta)), \tag{3}$$

where $h(s,\xi)$ is a bilinear form, not necessarily positive definite.

Define the space of closed loops

$$\Lambda = \{u = (r,x) \in H^1(S^1,\mathbb{R}) \times H^1(S^1,S^N)\} \tag{4}$$

Closed geodesics on (M, g_ε) are the critical points of $E_\varepsilon \colon \Lambda \to \mathbb{R}$ given by

$$E_\varepsilon(u) = \frac{1}{2}\int_0^1 g_\varepsilon(u)[\dot{u},\dot{u}]\,dt. \tag{5}$$

One has that

$$E_\varepsilon(u) = E_\varepsilon(r,x) = E_0(r,x) + \varepsilon G(r,x), \tag{6}$$

where

$$E_0(r,x) = \frac{1}{2}\int_0^1 \left(|\dot{r}|^2 + |\dot{x}|^2\right)\,dt$$

and

$$G(r,x) = \frac{1}{2}\int_0^1 h(r,x)[(\dot{r},\dot{x}),(\dot{r},\dot{x})]\,dt. \tag{7}$$

[1] Hereafter, we use the notation $\xi \bullet \eta = \sum_i \xi_i \eta_i$ for the scalar product in $\mathbb{R}^{N+1}$, and $|\xi|^2 = \xi \bullet \xi$.

In particular, we split E_0 into two parts, namely

$$E_0(r,x) = L_0(r) + E_{M_0}(x), \tag{8}$$

where

$$L_0(r) = \frac{1}{2}\int_0^1 |\dot{r}|^2\,dt, \quad E_{M_0}(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2\,dt.$$

The form of E_ε suggests to apply the perturbative results of [1] that we recall below for the reader's convenience.

Theorem 2.1 ([5, 1]). *Let H be a real Hilbert space, $E_\varepsilon \in C^2(H)$ be of the form*

$$E_\varepsilon(u) = E_0(u) + \varepsilon G(u), \tag{9}$$

where $G \in C^2(E)$.. Suppose that there exists a finite dimensional manifold Z such that

(AS1) $E_0'(z) = 0$ for all $z \in Z$;

(AS2) $E_0''(z)$ is a compact perturbation of the identity, for all $z \in Z$;

(AS3) $T_z Z = \ker E_0''(z)$ for all $z \in Z$.

There exist a positive number ε_0 and a smooth function $w\colon Z \times (-\varepsilon_0, \varepsilon_0) \to H$ such that the critical points of

$$\Phi_\varepsilon(z) = E_\varepsilon(z + w(z,\varepsilon)), \quad z \in Z, \tag{10}$$

are critical points of E_ε.

Moreover, it is possible to show that

$$\Phi_\varepsilon(z) = b + \varepsilon\Gamma(z) + o(\varepsilon), \tag{11}$$

where $b = E_0(z)$ and $\Gamma = G_{|Z}$. From this "first order" expansion, one infers

Theorem 2.2 ([1]). *Let H be a real Hilbert space, $E_\varepsilon \in C^2(H)$ be of the form* (9)*. Suppose that (AS1)–(AS3) hold. Then any strict local extremum of $G_{|Z}$ gives rise to a critical point of E_ε, for $|\varepsilon|$ sufficiently small.*

In the present situation, the critical points of E_0 are nothing but the great circles of S^N, namely

$$z_{p,q} = p\cos 2\pi t + q\sin 2\pi t, \tag{12}$$

where $p, q \in \mathbb{R}^{N+1}$, $p \bullet q = 0$, $|p| = |q| = 1$. Hence E_0 has a "critical manifold" given by

$$Z = \{z(r,p,q) = (r, z_{p,q}(\cdot)) \mid r \in \mathbb{R},\ z_{p,q} \text{ as in (12)}\}.$$

Lemma 2.3. *Z satisfies (AS2)–(AS3).*

Proof. The first assertion is known, see for instance [10].

For the second statement, we closely follow [6].

For $z \in Z$, of the form $z(t) = (r, z_{p,q}(t))$, it turns out that

$$E_0''(z)[h,k] = \int_0^1 \left[\dot{h} \bullet \dot{k} - |\dot{z}|^2 h \bullet k \right] dt$$

for any $h, k \in T_z Z$.

Let $e_i \in \mathbb{R}^{N+1}$, $i = 2, \ldots, N+1$, be orthonormal vectors such that $\{\frac{1}{2\pi}\dot{z}_{p,q}, e_2, \ldots e_{N+1}\}$ is a basis of $T_z Z$, and set

$$e_i(t) = \begin{cases} \dot{z}_{u^1,u^2}(t)/2\pi & \text{if } i = 1 \\ e_i & \text{if } i > 1, \end{cases}$$

Then, for h, k as before, we can write a "Fourier–type" expansion

$$h(t) = h_0(t)\frac{d}{dt} + \sum_{i=1}^{N-1} h_i(t)e_i(t), \quad k(t) = k_0(t)\frac{d}{dt} + \sum_{i=1}^{N-1} k_i(t)e_i(t). \tag{13}$$

Assume now that $h \in \ker E_0''(z_{p,q})$, i.e.

$$\int_0^1 \dot{h} \bullet \dot{k}\, dt = \int_0^1 |\dot{z}|^2 h \bullet k\, dt \quad \forall k \in T_{z_{p,q}} Z.$$

We plug (13) into this relations, and we get the system

$$\begin{cases} \ddot{h}_1 = 0 \\ \ddot{h}_j + 4\pi^2 h_j = 0 \quad j = 2, \ldots, N-1 \\ \ddot{h}_0 = 0. \end{cases} \tag{14}$$

Recalling that h_0 and h_1 are periodic, we find

$$\begin{cases} h_0 = \lambda_0, \quad h_1 = \lambda_1 \\ h_j = \lambda_j \cos 2\pi t + \mu_j \sin 2\pi t \quad j = 2, \ldots N-1. \end{cases} \tag{15}$$

Therefore, $h \in T_z Z$. This shows that $\ker E_0''(z_{p,q}) \subset T_{z_{p,q}} Z$. Since the converse inclusion is is always true, the lemma follows. $\square$

Lemma 2.4. *Suppose*

(h1) $h(r, \cdot) \to 0$ pointwise on S^N, as $|r| \to \infty$,

then

$$\Phi_\varepsilon \to b \equiv E_0(z).$$

Recall that Φ_ε was defined in (10).

Proof. This is proved as in [2, 5]. We just sketch the argument. The idea is to use the contraction mapping principle to characterize the function $w(\varepsilon, z)$ (see Theorem 1). Indeed, define

$$H(\alpha, w, z_r, \varepsilon) = \begin{pmatrix} E'_\varepsilon(z_r + w) - \alpha\dot{z} \\ w \bullet \dot{z}. \end{pmatrix}$$

So $H = 0$ if and only if $w \in (T_{z_r}Z)^\perp$ and $E'_\varepsilon(z_r + w) \in T_{z_r}Z$. Now,

$$H(\alpha, w, z_r, \varepsilon) = 0 \Leftrightarrow H(0, 0, z_r, 0) + \frac{\partial H}{\partial(\alpha, w)}(0, 0, z_r, 0)[\alpha, w] + R(\alpha, w, z_r, \varepsilon) = 0,$$

where $R(\alpha, w, z_r, \varepsilon) = H(\alpha, w, z_r, \varepsilon) - \frac{\partial H}{\partial(\alpha,w)}(0, 0, z_r, 0)[\alpha, w]$.

Setting

$$R_{z_r,\varepsilon}(\alpha, w) = -\left[\frac{\partial H}{\partial(\alpha, w)}(0, 0, z_r, 0)\right]^{-1} R(\alpha, w, z_r, \varepsilon),$$

one finds that

$$H(\alpha, w, z_r, \varepsilon) = 0 \Leftrightarrow (\alpha, w) = R_{z_r,\varepsilon}(\alpha, w).$$

By the Cauchy–Schwarz inequality, it turns out that $R_{zr,\varepsilon}$ is a contraction mapping from some ball $B_{\rho(\varepsilon)}$ into itself. If $|\varepsilon|$ is sufficiently small, we have proved the existence of (α, w) uniformly for $z_r \in Z$. We want to study the asymptotic behavior of $w = w(\varepsilon, z_r)$ as $|r| \to +\infty$. We denote by R^0_ε the functions $R_{z_r,\varepsilon}$ corresponding to the unperturbed energy functional $E_0 = E_{M_0}$. It is easy to see ([5], Lemma 3) that the function w^0 found with the same argument as before satisfies $\|w^0(z_r)\| \to 0$ as $|r| \to +\infty$. Thus, by the continuous dependence of $w(\varepsilon, z_r)$ on ε and the characterization of $w(\varepsilon, z_r)$ and w^0 as fixed points of contractive mappings, we deduce as in [2], proof of Lemma 3.2, that $\lim_{r\to\infty} w(\varepsilon, z_r) = 0$. In conclusion, we have that $\lim_{|r|\to+\infty} \Phi_\varepsilon(z_r + w(\varepsilon, z_r)) = E_{M_0}(z_0)$.

□

Remark 2.5. *There is a natural action of the group $O(2)$ on the space Λ, given by*

$$\begin{aligned} \{\pm 1\} \times S^1 \times \Lambda &\longrightarrow \Lambda \\ (\pm 1, \theta, u) &\mapsto u(\pm t + \theta), \end{aligned}$$

under which the energy E_ε is invariant. Since this is an isometric action under which Z is left unchanged, it easily follows that the function w constructed in Theorem 2.1 is invariant, too.

Theorem 2.6. *Assume that the functions $h_{ij} = h_{ji}$'s are smooth, bounded, and (h1) holds Then $M = \mathbb{R} \times M_0$ has at least N non-trivial closed geodesics, distinct modulo the action of the group $O(2)$. Furthermore, if*

(h2) *the matrix* $[h_{ij}(p,\cdot)]$ *representing the bilinear form* h *is positive definite for* $p \to +\infty$, *and negative definite for* $p \to -\infty$,

then M *possesses at least* $2N$ *non–trivial closed geodesics, geometrically distinct.*

Proof. Observe that $Z = \mathbb{R} \times Z_0$, where $Z_0 = \{z_{p,q} \mid |p| = |q| = 1,\ p \bullet q = 0\}$. According to Theorem 2.1, it suffices to look for critical points of Φ_ε. From Lemma 2.4, it follows that either $\Phi_\varepsilon = b$ everywhere, or has a critical point $(\bar{r}, \bar{p}, \bar{q})$. In any case such a critical point gives rise to a (non–trivial) closed geodesic of (M, g_ε).

From Remark 2.5, we know that Φ_ε is $O(2)$–invariant. This allows us to introduce the $O(2)$–category $\mathrm{cat}_{O(2)}$. One has

$$\mathrm{cat}_{O(2)}(Z) \geq \mathrm{cat}(Z/O(2)) \geq \mathrm{cuplength}(Z/O(2)) + 1.$$

Since $\mathrm{cuplength}(Z/O(2)) \geq N - 1$, (see [12]), then $\mathrm{cat}_{O(2)}(Z) \geq N$. Finally, by the Lusternik–Schnirel'man theory, M carries at least N closed geodesics, distinct modulo the action $O(2)$. This proves the first statement.

Next, let

$$\Gamma(r,p,q) = G((r, z_{p,q})) = \frac{1}{2}\int_0^1 h(r, z_{p,q}(t))[\dot{z}_{p,q}, \dot{z}_{p,q}]\,dt \tag{16}$$

Then (h) immediately implies that

$$\Gamma(r,p,q) \to 0 \quad \text{as } |r| \to \infty, \tag{17}$$

Moreover, if (h2) holds, then $\Gamma(r,p,q) > 0$ for $r > r_0$, and $\Gamma(r,p,q) < 0$ for $r < -r_0$. Since (recall equation (11))

$$\Phi_\varepsilon(r,p,q) = b + \varepsilon\Gamma(r,p,q) + o(\varepsilon), \tag{18}$$

it follows that

$$\begin{cases} \Phi_\varepsilon(r,p,q) > b & \text{for } r > r_0 \\ \Phi_\varepsilon(r,p,q) < b & \text{for } r < -r_0. \end{cases}$$

We can now exploit again the $O(2)$ invariance.

By assumption, and a simple continuity argument, $\{\Phi_\varepsilon > b\} \supset [R_0, \infty) \times Z_0$, and similarly $\{\Phi_\varepsilon < b\} \supset [-\infty, -R_0) \times Z_0$, for a suitably large $R_0 > 0$. Hence $\mathrm{cat}_{O(2)}(\{\Phi_\varepsilon > b\}) \geq \mathrm{cat}_{O(2)}(Z_0) = N$. The same argument applies to $\{\Phi_\varepsilon < b\}$. This proves the existence of at least $2N$ closed geodesics. $\square$

Remark 2.7. *(i) In [6], the existence of* N *closed geodesics on* S^N *endowed with a metric close to the standard one is proved. Such a result does not need any study of* Φ_ε *and its behavior. The existence of* $2N$ *geodesics is, as far as we know, new. We emphasize that it strongly depends on the* form *of* $M = \mathbb{R} \times M_0$.

(ii) In [13], the metric g *on* M *is possibly not perturbative. No multiplicity result is given.*

3 The general case

In this section we consider a compact riemannian manifold (M_0, g_0), and in analogy to the previous section, we put

$$g_\varepsilon(s,\xi)((r,\eta),(r,\eta)) = |r|^2 + g_0(\xi)(\eta,\eta) + \varepsilon h(s,\xi)((r,\eta),(r,\eta)). \tag{19}$$

Again, we define $\Lambda = \{u = (r,x) \mid r \in H^1(S^1,\mathbb{R}),\ x \in H^1(S^1, M_0)\}$,

$$E_{M_0}(x) = \frac{1}{2}\int_0^1 g_0(x)(\dot{x},\dot{x})\,dt, \quad E_0(r,x) = \frac{1}{2}\int_0^1 |\dot{r}|^2\,dt + E_{M_0}(x),$$

and finally

$$E_\varepsilon(r,x) = E_0(r,x) + \varepsilon G(r,x),$$

with G as in (7). It is well known ([11]) that M_0 has a closed geodesic z_0. The functional E_{M_0} has again a critical manifold Z given by

$$Z = \{u(\cdot) = (\rho, z_0(\cdot + \tau)) \mid \rho \text{ constant}, \tau \in S^1\}.$$

Let $Z_0 = \{z_0(\cdot + \tau) \mid \tau \in S^1\}$. It follows that $Z \approx \mathbb{R} \times Z_0$. The counterpart of Γ in (11) is

$$\Gamma(r,\tau) = \frac{1}{2}\int_0^1 h(r, z_\tau)[\dot{z}_\tau, \dot{z}_\tau]\,dt. \tag{20}$$

Let us recall some facts from [10].

Remark 3.1. *There is a linear operator $A_z \colon T_z\Lambda(M_0) \to T_z\Lambda$, which is a compact perturbation of the identity, such that*

$$E''_{M_0}(z)[h,k] = \langle A_z h \mid k\rangle_1 = \int_0^1 \overset{\cdot}{\overbrace{A_z h}} \bullet \dot{k}\,dt.$$

In particular, E_0 satisfies (AS2).

Definition 3.2. *Let*

$$\ker E''_{M_0}(z_0) = \{h \in T_{z_0}\Lambda(M_0) \mid \langle A_{z_0} h \mid k\rangle_1 = 0 \quad \forall k \in T_{z_0}\Lambda(M_0)\}.$$

We say that a closed geodesic z_0 of M_0 is non–degenerate*, if*

$$\dim\ker E''_{M_0}(z_0) = 1.$$

Remark 3.3. *For example, it is known that when M_0 has negative sectional curvature, then all the geodesics of M_0 are non–degenerate. See [7]. Moreover, it is easy to see that the existence of non–degenerate closed geodesics is a* generic *property.*

Lemma 3.4. *If z_0 is a non–degenerate closed geodesic of M_0, then Z satisfies (AS2).*

Proof. It is always true that $T_{z_r}Z \subset \ker E_0''(z_r)$. By (26), we have that $\dim T_{z_r}Z = \dim\ker E_0''(z_r)$. This implies that $T_{z_r}Z = \ker E_0''(z_r)$. A generic element of Z has the form (ρ, z^τ) for $\rho \in \mathbb{R}$ and $z^\tau = z(\cdot + \tau)$; then

$$T_{(\rho,z^\tau)}M = \mathbb{R} \times T_{z^\tau}M_0,$$

and any two vector fields Y and W along a curve on $M = \mathbb{R} \times M_0$ can be decomposed into

$$Y = h(t)\frac{d}{dt} + y(t) \in \mathbb{R} \oplus T_{z^\tau}Z_0, \tag{21}$$

$$W = k(t)\frac{d}{dt} + w(t) \in \mathbb{R} \oplus T_{z^\tau}Z_0. \tag{22}$$

In addition, there results (see [9])

$$E''_{M_0}(z_0)[y,w] = \int_0^1 \left[g_0(D_t y, D_t w) - g_0(R_{M_0}y(t), \dot z_0(t))\dot z_0(t) \mid w(t))\right] dt, \tag{23}$$

and

$$R_M(r,z) = R_{\mathbb{R}}(r) + R_{M_0}(z) = R_{M_0}(z), \tag{24}$$

where R_M, R_{M_0}, etc. stand for the curvature tensors of M, M_0, etc. By (23), (21) and (22), as in the previous section, $E_0''(\rho, z_\tau)[Y,W] = 0$ is equivalent to the system

$$\begin{cases} \ddot h = 0 \\ \int_0^1 g_0(z)[D_t y, D_t w] - \langle R_{M_0}(y(t), \dot z_r(t))\dot z_r(t) \mid w(t)\rangle\, dt = 0. \end{cases} \tag{25}$$

As in the case of the sphere, the first equation implies that h is constant. The second equation in (25) implies that $y \in \ker E''_{M_0}(z^\tau) = \ker E''_{M_0}(z_0)$. Hence,

$$\ker E_0''(z_r) = \{(h,y) \mid h \text{ is constant, and } y \in \ker E''_{M_0}(z_0)\}. \tag{26}$$

This completes the proof. □

Theorem 3.5. *Let M_0 be a compact, connected manifold of dimension $N < \infty$. Assume that M_0 admits a non–degenerate closed geodesic z, and that (in local coordinates) $h_{ij}(p,\cdot) \to a_-$ as $p \to -\infty$, and $h_{ij}(p,\cdot) \to a_+$ as $p \to +\infty$.*

1. *If $a_- = a_+$ and $h_{ij}(p,\cdot)$ satisfies (h2), then M has at least one closed geodesic.*
2. *If $a_- \le a_+$ and $h_{ij}(p,\cdot)[u,v] - a(u \mid v)$ is negative definite for $p \to -\infty$ and positive definite for $p \to +\infty$, then M has at least two non-trivial closed geodesic.*

Proof. Lemma 3.4 allows us to repeat all the argument in Theorem 2.6, and the result follows immediately. □

4 Isolated geodesics

In this final section, we discuss one situaion where the critical manifold Z may be degenerate. Here, the non–degeneracy condition (AS3) fails, and $T_z Z \subset \ker E_0''(z)$ strictly. Fix a closed geodesic Z_0 for M_0, and put $\tilde{W} = (T_{z_0} Z)^\perp$. Since $T_z Z \subset \ker E_0''(z)$ strictly, there exists $k > 0$ such that $\tilde{W} = (\ker E_0''(z_0))^\perp \oplus \mathbb{R}^k$. Repeating the preceding finite dimensional reduction, one can find again a unique map $\tilde{w} = \tilde{w}(z, \zeta)$, where $z \in Z$ and $\zeta \in \mathbb{R}^k$, in such a way that $E_\varepsilon' = 0$ reduces to an equation like

$$\nabla A(z + \zeta + \tilde{w}(z, \zeta)) = 0.$$

If z_0 is an isolated minimum of the energy E_{M_0} over some connected component of $\Lambda(M_0)$, then it is possible to show that there exists again a function $\Gamma \colon Z \to \mathbb{R}$ such that

$$\nabla A(z + \zeta + \tilde{w}(z, \zeta)) = 0 \iff \frac{\partial \Gamma}{\partial r}(-R, \tau) \frac{\partial \Gamma}{\partial r}(R, \tau) \neq 0$$

for some $R \in \mathbb{R}$ and all $\tau \in S^1$. For more details, see [4]. In particular, we will use the following result.

Theorem 4.1. *Let H be a real Hilbert space, $f_\varepsilon \colon H \to \mathbb{R}$ is a family of C^2–functionals of the form $f_\varepsilon = f_0 + \varepsilon G$, and that:*

(f0) f_0 has a finite dimensional manifold Z of critical points, each of them being a minimum of f_0;

(f1) for all $z \in Z$, $f_0''(z)$ is a compact perturbation of the identity.

Fix $z_0 \in Z$, put $W = (T_{z_0} Z)^\perp$, and suppose that $(f_0)_{|W}$ has an isolated minimum at z_0. Then, for ε sufficiently small, f_ε has a critical point, provided $\deg(\Gamma', B_R, 0) \neq 0$.

Remark 4.2. *Theorem 4.1 has been presented in a linear setting. For Riemannian manifold, we can either reduce to a* local *situation and then apply the exponential map, or directly resort to the slightly more general degree theory on Banach manifold developed in [8].*

Theorem 4.3. *Assume that $\pi_1(M_0) \neq \{0\}$, and that all the critical points of E_0, the energy functional of M_0, are isolated. Suppose the bilinear form h satisfies (h1), and*

(h3) $\dfrac{\partial h}{\partial r}(R, \xi) \dfrac{\partial h}{\partial r}(-R, \xi) \neq 0$ *for some $R > 0$ and all $\xi \in S^1$.*

Then, for $\varepsilon > 0$ sufficiently small, the manifold $M = \mathbb{R} \times M_0$ carries at least one closed geodesic.

Proof. We wish to use Theorem 4.1. Since $\pi_1(M_0) \neq \{0\}$, then E_0 has a geodesic z_0 such that $E_0(z_0) = \min E_0$ over some component C of $\Lambda(M_0)$. See [11].

We consider the manifold

$$Z = \{u \in \Lambda \mid u(t) = (\rho, z_0(t+\tau)),\ \rho \text{ constant}, \tau \in S^1\}.$$

Here we do not know, a priori, if Z is non–degenerate in the sense of condition (AS2). But of course $(E_0)_W$ has a minimum at the point (ρ, z_0), where $W = (T_{\rho,z^\tau})Z)^\perp$. We now check that it is isolated for $(E_0)_W$. We still know that $Z = \mathbb{R} \times Z_0$. Take any point $(\rho, z_\tau) \in Z$, and observe that $T_{(\rho,z^\tau)}Z = \{(r,y) \mid r \in \mathbb{R},\ y \in T_{z^\tau}Z_0\}$. For all $(r,y) \in W$ sufficiently close to (ρ, z_0), it holds in particular that $y \perp z_\tau$. Hence

$$E_0(r,y) = L_0(r) + E_{M_0}(y) \geq E_{M_0}(y) > E_{M_0}(z^\tau) = E_{M_0}(z_0) = E_0(\rho, z_0)$$

since $L_0 \geq 0$ and z_0 (and hence z^τ, due to $O(2)$ invariance) is an isolated minimum of E_{M_0} by assumption.

Finally, thanks to assumption (h3), $\frac{\partial \Gamma}{\partial r}(-R,\tau)\frac{\partial \Gamma}{\partial r}(R,\tau) \neq 0$.

This concludes the proof. ∎

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